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Reynolds-number-independent instability of the boundary layer over a flat surface

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A three-dimensional mode of spatial instability, related to the temporal algebraic growth that determines lift-up in parallel flow, is found to occur in the two-dimensional boundary layer growing over a flat surface. This unstable perturbation can be framed within the limits of Prandtl's standard boundary-layer approximation, and therefore develops at any Reynolds number for which the boundary layer exists, in sharp contrast to all previously known flow instabilities which only occur beyond a sharply defined Reynolds-number threshold. It is thus a good candidate for the initial linear amplification mechanism that leads to bypass transition.

1. Introduction

The stability or instability of a physical system may be judged differently according to the mathematical model that is used to describe it. Whereas an ideally infinite fluid boundary layer sooner or later becomes unstable according to the Navier–Stokes equations, ever since Stewartson's (1957) and Libby & Fox's (1964) linear analyses of small perturbations to the self-similar two-dimensional boundary layer common wisdom has had it that the boundary layer as described by the Prandtl equations is intrinsically stable; additional effects, which are present in the full Navier–Stokes model but not in Prandtl's boundary-layer equations and become non-negligible above a certain Reynolds-number threshold, have always been considered to be the only possible cause of instability. Examples of such effects are the pressure coupling of short-longitudinal-wavelength two-dimensional perturbations (Tollmien–Schlichting waves) and the centrifugal coupling of spanwise-varying perturbations over a concave wall (Görtler vortices). Neither exists within the context of Prandtl's boundary-layer theory, but only in appropriate extensions of it; as far as Prandtl's equations are valid, the complete set of eigenmodes determined by Libby & Fox have a negative-power dependence on the streamwise coordinate x, the least damped of them dying out as x^{-1}. (Stewartson had found an analytical expression of this dominant mode and a physical explanation of why the associated exponent is -1 exactly.) However, Stewartson's and Libby & Fox's analyses are completely two-dimensional, and presume a two-dimensional perturbation.

Recent developments in the study of the so-called 'algebraic growth' (e.g. Landahl 1980; Hultgren & Gustavsson 1981; Boberg & Brosa 1988; Gustavsson 1991; Butler & Farrell 1992; Reddy & Henningson 1993; Trefethen *et al.* 1993) have shed light on the fact that the response of a two-dimensional flow to three-dimensional perturbations can be even qualitatively different from its response to two-dimensional ones. By a phenomenon of resonance between the Rayleigh and Squire equations, a spanwise-oscillating perturbation can arise which exhibits a linear growth in time even when an eigenvalue analysis would indicate stability (Ellingsen & Palm 1975). Once viscosity is put back into the model, a slow exponential decay superposes over the linear growth, so that the perturbation must eventually die out; the combined outcome is that, as remarked by Hultgren & Gustavsson (1981), the inviscid algebraic growth prevails for a possibly long but finite intermediate stage, and after this stage the asymptotic viscous decay sets in (in parallel flow, that is).

However, what happens when a phenomenon of algebraic growth takes place inside a spatially broadening boundary layer? The damping effect of viscosity now weakens with distance, and it is not *a priori* evident whether it can still eventually overcome the algebraic growth. This is the question that the present paper originated from.

2. Libby–Fox–Stewartson theory

Let us consider boundary-layer flow over a flat plate, governed by the usual steady two-dimensional Prandtl equation, i.e. in streamfunction form

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \psi_{yyy}, \tag{1}$$

Blasius' similarity solution, $\psi = x^{1/2} F_0(y/x^{1/2})$, obeys the well-known ordinary differential equation

$$F_{0,\eta\eta\eta} + \tfrac{1}{2} F_0 F_{0,\eta\eta} = 0, \tag{2}$$

where $\eta = y/x^{1/2}$ and the boundary conditions are $F_0(0) = F_{0,\eta}(0) = 0$, $F_{0,\eta}(\infty) = 1$. If we change variables from $\psi(x, y)$ to $F(x, \eta) = \psi/x^{1/2}$, without assuming F necessarily to be a function of η only, we obtain

$$x F_\eta F_{x\eta} - x F_{\eta\eta} F_x = F_{\eta\eta\eta} + \tfrac{1}{2} F F_{\eta\eta} \tag{3}$$

which is completely equivalent to (1) and, of course, admits (2) as a particular case. Libby & Fox (1964) sought solutions of the form $F = F_0(\eta) + \delta F(x, \eta)$ by linearizing (3) with respect to the small perturbation δF. The linearized equation turns out to admit eigensolutions of the form $\delta F = x^{-r} N(\eta)$, with $N(\eta)$ and r determined by the following eigenvalue problem:

$$N_{\eta\eta\eta} + \tfrac{1}{2} F_0 N_{\eta\eta} + \tfrac{1}{2} F_{0,\eta\eta} N + r(F_{0,\eta} N_\eta - F_{0,\eta\eta} N) = 0; \tag{4}$$

$N(0) = N_\eta(0) = N_\eta(\infty) = 0$. One of the eigensolutions can be determined analytically, as discovered by Stewartson (1957), if advantage is taken of the translational symmetry of the original equation (1). In particular, $\psi = (x+\epsilon)^{1/2} F_0[y/(x+\epsilon)^{1/2}]$ represents another exact solution of (1) (corresponding to a different initial condition); by linearizing with respect to ϵ one finds $\psi/x^{1/2} = F_0(y/x^{1/2}) + \epsilon[F_0(\eta) - \eta F_{0,\eta}(\eta)]/2x$. Therefore $N_1 = F_0(\eta) - \eta F_{0,\eta}(\eta)$ is an exact solution of (4) corresponding to the eigenvalue $r_1 = 1$, as can also be independently verified.The first ten eigenvalues were numerically determined by Libby & Fox to be 1, 1.887, 2.817, 3.800, 4.740, 5.6, 6.6, 7.5, 8.4, 9.3 (more accurate values are given here in table 1). Libby & Fox were also able to transform (4) into a self-adjoint problem, by defining an auxiliary variable H as $H = (N/F_{0,\eta})_\eta$. It turns out that H obeys a second-order Sturm–Liouville equation, whence follows that an infinite discrete sequence of positive real eigenvalues r_n exists and that the set of eigenmodes is complete.

Although the calculation of eigenmode amplitudes from initial conditions imposed at non-zero x is discussed at length in Libby & Fox (1964), they do not seem to have realized that the limit for $x \to 0$ of the left eigenfunctions of their equation exists, and therefore the mode amplitudes can be calculated from incident-stream initial conditions given at the leading edge. We give this result in Appendix A.

k	r_k	C_k	c_k
1	1	9.069301518	3.01152810
2	1.886820416	4.591899002	−0.57122779
3	2.814356780	3.259806394	0.067952251
4	3.756605180	2.590863034	-5.811340×10^{-3}
5	4.707176465	2.180249834	3.85743×10^{-4}
6	5.663269971	1.899019184	-2.0861×10^{-5}
7	6.623387663	1.692513402	9.500×10^{-7}
8	7.586856793	1.532421057	-3.73×10^{-8}
9	8.555317860	1.397265714	1.28×10^{-9}
10	9.539710607	1.259676434	-3.5×10^{-11}

TABLE 1. Parameters of the Libby & Fox modes

3. Three-dimensional boundary-layer perturbations

One of two different three-dimensional formulations of the boundary-layer approximation can generally be adopted, depending on whether the spanwise scale of the phenomenon being considered is comparable to the longitudinal scale L or to the normal scale $\delta=(\nu L/U)^{1/2}$ (where ν denotes the kinematic viscosity and U the dimensional value of the outer velocity). The first case is typically produced by a three-dimensional outer stream, and is governed by the equations (e.g. Schlichting 1968, p. 239)

$$u_x+v_y+w_z=0, \tag{5a}$$

$$uu_x+vu_y+wu_z=\tfrac{1}{2}(U^2+W^2)_x+u_{yy}, \tag{5b}$$

$$uw_x+vw_y+ww_z=\tfrac{1}{2}(U^2+W^2)_z+w_{yy}, \tag{5c}$$

with boundary conditions $u(x,0,z)=v(x,0,z)=w(x,0,z)=0$, $u(x,\infty,z)=U(x,z)=\Phi_x$, $w(x,\infty,z)=W(x,z)=\Phi_z$ (where Φ is the potential at the wall of the imposed irrotational outer stream). The second case is typical of three-dimensionalities originating inside the boundary layer itself, and leads to the equations (used e.g. by Hall 1983 for the study of the Görtler instability, and by Luchini 1995 and Luchini & Trombetta 1995 for the boundary layer over a grooved surface)

$$u_x+v_y+w_z=0, \tag{6a}$$

$$uu_x+vu_y+wu_z=UU_x+u_{yy}+u_{zz}, \tag{6b}$$

$$uv_x+vv_y+wv_z+p_y=v_{yy}+v_{zz}, \tag{6c}$$

$$uw_x+vw_y+ww_z+p_z=w_{yy}+w_{zz}, \tag{6d}$$

with boundary conditions $u(x,0,z)=v(x,0,z)=w(x,0,z)=0$, $u(x,\infty,z)=U(x)$, $w(x,\infty,z)=0$, $p(x,\infty,z)$. Both systems of equations (5) and (6) are parabolic. Neither allows v to vanish at infinity.

If (5) and (6) are linearized in a neighbourhood of the two-dimensional Blasius solution $u_0=F_{0,\eta}(y/x^{1/2})$, and the perturbation is assumed to have the complex-exponential form $u=u_0+\delta u(x,y)\,e^{i\alpha z}$, they respectively become

$$\delta u_x+\delta v_y+i\alpha\delta w=0, \tag{7a}$$

$$u_0\,\delta u_x+v_0\,\delta u_y+u_{0,x}\,\delta u+u_{0,y}\,\delta v=\delta u_{yy}, \tag{7b}$$

$$u_0\,\delta w_x+v_0\,\delta w_y=\delta w_{yy}, \tag{7c}$$

and

$$\delta u_x + \delta v_y + \mathrm{i}\alpha \delta w = 0, \tag{8a}$$

$$u_0\, \delta u_x + v_0\, \delta u_y + u_{0,x}\, \delta u + u_{0,y}\, \delta v = \delta u_{yy} - \alpha^2 \delta u, \tag{8b}$$

$$u_0\, \delta v_x + v_0\, \delta v_y + v_{0,x}\, \delta u + v_{0,y}\, \delta v + \delta p_y = \delta v_{yy} - \alpha^2 \delta v, \tag{8c}$$

$$u_0\, \delta w_x + v_0\, \delta w_y + \mathrm{i}\alpha \delta p = \delta w_{yy} - \alpha^2 \delta w. \tag{8d}$$

Equations (8) appear to offer the most general context in which to study the evolution of three-dimensional perturbations imposed over the Blasius flat-plate boundary layer. Incidentally they are equivalent, apart from the absence of any curvature, to Hall's (1983) formulation of the Görtler problem. However, these equations offer no self-similarity, mainly because the two terms in the sum $\delta u_{yy} - \alpha^2 \delta u$ scale differently: the first term is proportional, when expressed in similarity variables, to x^{-1} and the second to x^0. In other words, the normal scale of the perturbation expands with increasing x whereas the spanwise scale remains constant. Because of this difficulty, an extension of the Libby & Fox theory to (8) is impossible. The only two possibilities, which have emerged in the context of the Görtler problem, are either to adopt a quasi-parallel approximation (the route originally followed by Görtler himself and later perfected by many others; see e.g. Bottaro & Luchini 1995) or to resort to a numerical solution of the parabolic equations as proposed by Hall. However, the quasi-parallel approach can only be theoretically justified for a Görtler number larger than unity (Bottaro & Luchini 1995), and therefore certainly not for the present case which corresponds to a Görtler number of zero; the numerical approach is useful to explore the effects of varying initial conditions, but fails to give a general indication as to the eventual behaviour of the perturbation far from its source.

However, there is a third choice: to look for a special wavenumber range in which the spanwise and normal scales are different enough that the structure of the equations simplifies and the determination of self-similar eigenmodes in Libby & Fox fashion becomes possible. This range is that of small wavenumber α, in the non-dimensionalization appropriate to (8) where the reference z-scale is δ, or more precisely the range $1/L \leqslant \alpha \ll 1/\delta$ in dimensional form. At first it may appear that the small-α range should trivially coincide with $\alpha = 0$, that is with two-dimensional behaviour. But this is only partially true. If the small-α limit of (8) is taken while keeping the product $\alpha \delta w$ of order unity, the result is (7), and for small but non-zero α these are *not* equivalent to the two-dimensional problem. For, even if α is small, δw can be large enough to make the term $\mathrm{i}\alpha w$ non-negligible in the continuity equation; indeed (7) are obtained in the case of α of order $1/L$ and w of order U, whereas (8) refer to the case of α of order $1/\delta$ and w of order $(\delta/L)\,U$. Any intermediate case with $1/L \leqslant \alpha \ll 1/\delta$ and δw of order $(\alpha L)^{-1} \delta u$ is still correctly represented by (7).

4. Self-similar three-dimensional perturbations

Within the system of equations (7), (7c) can be solved independently. On seeking a solution of the Libby & Fox form $\delta w = x^{-s} h(\eta)$ the problem reduces to the ordinary differential equation

$$h_{\eta\eta} + F_0 h_\eta / 2 + s F_{0,\eta} h = 0, \tag{9}$$

with the boundary conditions $h(0) = 0$, $h(\infty) = 0$. Equation (9) can be solved numerically with ease. The first few eigenvalues of s are thus determined to be as given

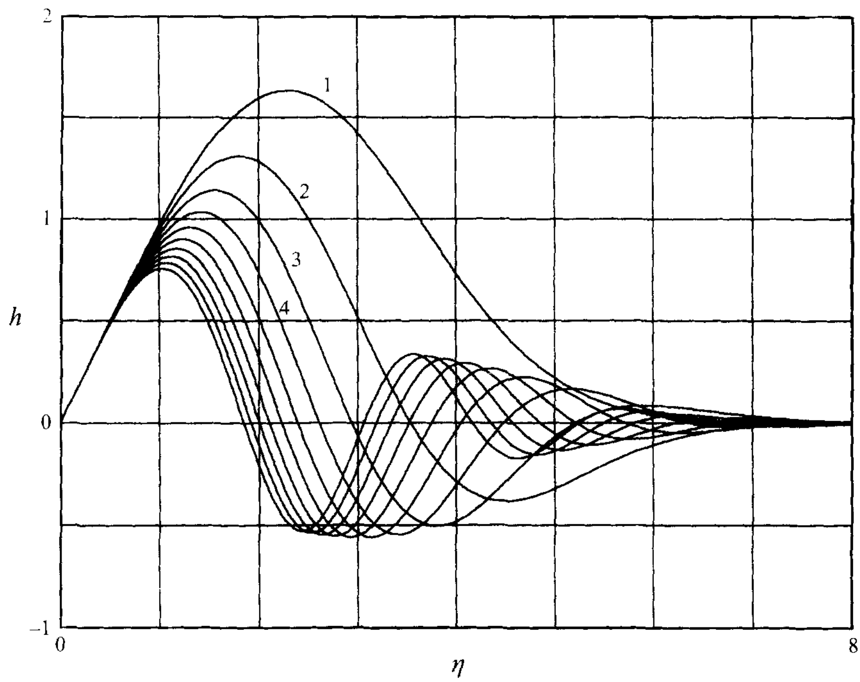

FIGURE 1. Spanwise-velocity profiles $h(\eta)$ for the first ten eigenfunctions of the three-dimensional perturbation equation (9).

k	s_k	D_k	d_k
1	0.786565917	29.277354800	6.84949861
2	1.693645408	20.509398414	−1.72478993
3	2.627241252	17.149057187	0.23986893
4	3.572715439	15.220410887	−0.02281352
5	4.525352271	13.920715467	1.64119×10^{-3}
6	5.482911661	12.963900452	-9.4690×10^{-5}
7	6.444131016	12.218500202	4.552×10^{-6}
8	7.408368052	11.609771265	-1.87×10^{-7}
9	8.376653746	11.060368540	6.6×10^{-9}
10	9.357475196	10.412698579	-2.0×10^{-10}

TABLE 2. Parameters of the three-dimensional modes

in table 2. The corresponding eigenfunctions, normalized with the condition $h_\eta(0) = 1$, are plotted in figure 1.

Once (7c) is solved, the solutions of (7a, b) can be divided up in two classes: one has $\delta w = 0$ and contains the two-dimensional modes calculated by Libby & Fox; the other has each of the modes of (9) in turn as excitation, and corresponds in the inviscid limit to the perturbations that exhibit algebraic growth. On lettering $\delta u = \mathrm{i}\alpha x^{-r} g(\eta)$ and $\delta y = \mathrm{i}\alpha x^{-r-1/2}[\eta g(\eta)/2 + (r-\frac{1}{2}) f(\eta)]$ equations (7a, b) become

$$g = f_\eta + h/(r-\tfrac{1}{2}), \qquad (10a)$$

$$g_{\eta\eta} + \tfrac{1}{2}F_0 g_\eta + rF_{0,\eta} g - (r-\tfrac{1}{2}) F_{0,\eta\eta} f = 0, \qquad (10b)$$

where similarity requires that $r = s-1$. A few numerical solutions of (10) with the

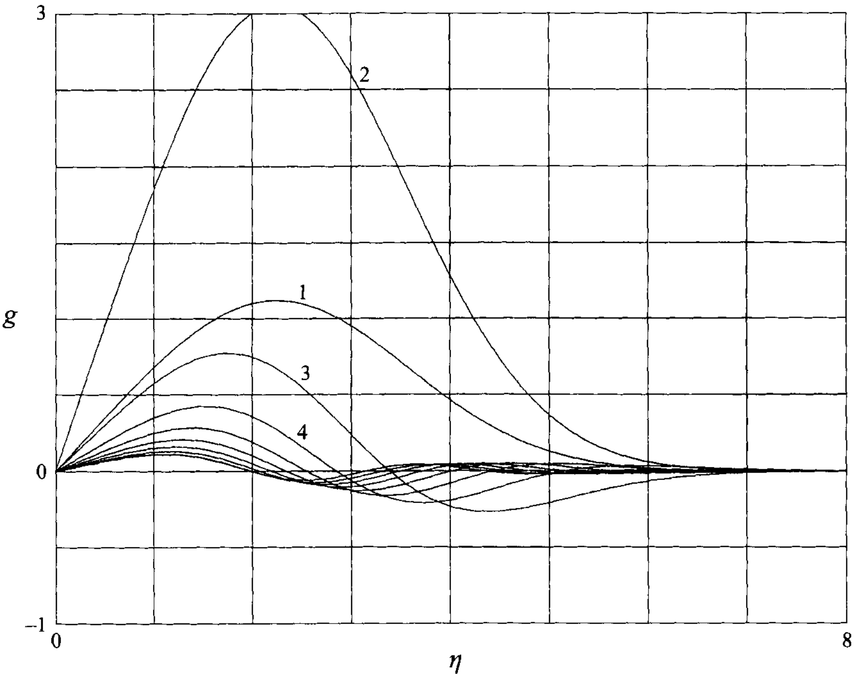

FIGURE 2. Longitudinal-velocity profiles $g(\eta)$ for the first ten eigenfunctions of the three-dimensional perturbation, equations (10).

boundary conditions $f(0) = g(0) = 0$, $g(\infty) = 0$ are plotted in figure 2. We thus obtain the key result that the first s-eigenvalue, namely 0.787, is less than 1, and therefore, the first r-eigenvalue, namely -0.213, is *negative*. Therefore, while the w-component of this particular perturbation decays proportionally to $x^{-0.787}$, its u-component *grows unboundedly* as $x^{+0.213}$.

5. Excitation of the instability

In order to expand an arbitrary initial spanwise velocity profile into a sum of modes, we can cast (9) in a Sturm–Liouville form through a procedure similar to the one followed by Libby & Fox. On replacing $\frac{1}{2}F_0$ by $-F_{0,\eta\eta\eta}/F_{0,\eta\eta}$, according to (2), we can easily rewrite (9) as

$$(h_\eta/F_{0,\eta\eta})_\eta + s(F_{0,\eta}/F_{0,\eta\eta})\,h = 0. \tag{11}$$

It follows that the eigenvalues are real and positive, and that the eigenfunctions form a complete set and are mutually orthogonal with weight $(F_{0,\eta}/F_{0,\eta\eta})$. Therefore an arbitrary disturbance δw can be represented as

$$\delta w = \sum_{k=1}^{\infty} b_k\, x^{-s_k}\, h_k(\eta), \tag{12}$$

where, analogously to the Libby & Fox case reported in Appendix A, equation (A 6),

$$b_k = x_i^{s_k} D_k^{-1} \int_0^\infty \delta w(x_i, y)\, h_k(\eta)\,(F_0'/F_0'')\,\mathrm{d}\eta \tag{13}$$

and

$$D_k = \int_0^\infty h_k^2 (F_0'/F_0'')\,\mathrm{d}\eta. \tag{14}$$

Most often one will be interested in calculating the effect of an initial perturbation carried along by the oncoming stream, and therefore known at the leading edge $x_i = 0$ of the plate. When this is the case, (13) can be cast in a form valid at $x_i = 0$ by changing the integration variable from η to y, so as to obtain

$$b_k = D_k^{-1} \int_0^\infty \delta w(x_i, y)\, x_i^{(s_k - 1/2)}\, h_k(y/x_i^{1/2})\,(F_0'/F_0'')\,\mathrm{d}y \tag{15}$$

and noticing that η becomes infinitely large in the limit $x_i \to 0$. The large-η behaviour of $h_k(\eta)$ can be obtained from a WKB asymptotic analysis of (9), and is given by $h_k \approx d_k \eta^{(2s_k-1)} F_0''(\eta)$, where d_k is a coefficient which can be estimated numerically (Appendix B). Therefore (15) for $x_i = 0$ becomes

$$b_k = \frac{d_k}{D_k} \int_0^\infty \delta w(0, y)\, y^{(2s_k - 1)}\, \mathrm{d}y. \tag{16}$$

The first ten values of d_k and D_k are given in table 2.

6. General order-of-magnitude considerations

It may be tempting to dismiss the instability just described as one that can hardly be observed in practice since, after all, it only grows as $x^{0.213}$ which is very slow, but in fact its consequences are very real. The key point here is not so much how fast the perturbation grows, but rather that it is *not damped*. In fact, the two-dimensional perturbations studied by Stewartson and Libby & Fox decay at least as x^{-1}, and therefore tend to disappear in the boundary layer. When a spanwise-velocity perturbation hits the leading edge, it is decomposed into a sum of modes of (9) which decay at least as $x^{-0.787}$, and in this sense do not behave very differently from the two-dimensional case. At the same time, however, each of these modes is accompanied by the corresponding driven mode of (10), the first of which is not damped. If no perturbation of the longitudinal velocity is present at the leading edge, the longitudinal-velocity profile associated with these modes must be exactly compensated by a suitable superposition of Libby & Fox modes, which certainly exists since these modes form a complete set, such that the initial longitudinal velocity perturbation is identically zero. Further down, however, the Libby & Fox modes decay, freeing the slowly growing unstable mode to emerge in its full amplitude, which can be larger than that of the driving spanwise perturbation.

To estimate the order of magnitude of the longitudinal-velocity perturbations involved, we can start from (16). This equation says that a longitudinal perturbation of order unity, that is comparable to the basic velocity profile and thus capable of driving the boundary layer out of the linear small-perturbation regime, is generated by a spanwise perturbation such that $\alpha \delta w y^{1.57} = O(1)$. On restoring variables in their physical dimensions, and using δ for the typical y-scale, this translates to

$$w\delta/\nu = O(\alpha\delta)^{-1}. \tag{17}$$

(The same result could be arrived at by inspection of the continuity equation (7*a*): once a mechanism is established that couples the transverse and longitudinal components of velocity, αw must be comparable in order of magnitude to U/L, which by definition equals ν/δ^2.)

There are two possible physical interpretations of (17). In one, $w\delta/\nu$ can be read as a perturbation Reynolds number. Therefore, the first interpretation is that instability occurs whenever the Reynolds number *of the perturbation* exceeds $(\alpha\delta)^{-1}$. Since the minimum value of $(\alpha\delta)^{-1}$ is approached near the end of the permitted α range where $\alpha \approx 1/\delta$, and is unity, the earliest instability will occur at this end of the range. Therefore, we can say that the boundary layer becomes unstable when the Reynolds number *of the perturbation* becomes large compared to unity, independent of the Reynolds number of the *unperturbed flow*.

From a slightly different viewpoint if α has its most unstable value, i.e. it is not far from $1/\delta$, the product $w\delta \approx w/\alpha$ in (17) can be interpreted as the order of magnitude of the circulation of the oncoming longitudinal vortex; therefore, an easy to remember, if slightly dramatic, interpretation of the result is that an externally introduced longitudinal vortex with a circulation somewhat larger than ν can, if its size is comparable to the boundary-layer thickness, permanently disrupt a flat-plate Blasius boundary layer.

7. Connection with bypass transition and temporal algebraic growth theories

After many years of being led by Squire's theorem to focus their attention onto a two-dimensional route to transition mediated by Tollmien–Schlichting waves, researchers in fluid mechanics today agree that an alternative mechanism exists: the so-called bypass transition. For instance, the recent review by Kachanov (1994), which is otherwise committed to the secondary instabilities and nonlinear breakdown of Tollmien–Schlichting waves, begins by saying:

> The physical mechanisms of the transition phenomenon depend essentially on the specific type of flow and on the character of environmental disturbances. For boundary-layer flows two main classes of transition are known (Morkovin 1968, 1984; Morkovin and Reshotko 1990). The first of them is connected with boundary-layer instabilities (described initially by linear stability theories), amplification and interaction of different instability modes resulting in the laminar flow breakdown. This class is usually observed when environmental disturbances are rather small. The second class of transition, usually called bypass, is connected with 'direct' nonlinear laminar-flow breakdown under the influence of external disturbances. This is observed when high enough levels of environmental perturbations (free-stream disturbances, surface roughness, etc.) are present.

In fact, bypass transition has for a long time been something that experimentalists observed but theoreticians could not explain. Experiments since those of Klebanoff, Tidstrom & Sargent (1962) have displayed features, in particular longitudinal low-velocity streaks, which are completely extraneous to the linear theory of two-dimensional Tollmien–Schlichting waves. Only under very low levels of free-stream turbulence, as are probably present in the high atmosphere but are very hard to reproduce in a wind tunnel, can the linear evolution of Tollmien–Schlichting waves be observed. Actually, in order to overcome environmental disturbances, in most experiments Tollmien–Schlichting waves are artificially generated by a periodic excitation of appropriate frequency. Thus for a long time experimentalists have been seeing streaks and theoreticians have been predicting Tollmien–Schlichting waves, ascribing the mysterious bypass transition to something that escaped linear small-perturbation theory and with it Squire's theorem. The above except from Kachanov (1994) still reflects the common view that an unspecified 'direct' nonlinear mechanism driven by large values of environmental disturbances is at the origin of bypass transition.

On the other hand, a linear mechanism likely to be responsible for the Klebanoff streaks has been emerging during the 1980s and 1990s under the names of *lift-up* and *algebraic growth*.

Initially, a physical mechanism was identified according to which a longitudinal externally generated vortex grazing the wall would lift up low-velocity fluid on one side and drive down high-velocity fluid on the other, thus creating a streak-like spanwise non-uniformity in the near-wall velocity; it was thus gradually recognized that the exponential growth of a single dominant mode was not the only relevant amplification mechanism provided by linear theory. In particular, Ellingsen & Palm (1975) and Landahl (1980) analysed the lift-up mechanism in the context of inviscid-flow theory and found that if the flow field has no streamwise variation (Ellingsen & Palm) or at least contains a Fourier component with zero streamwise wavenumber (Landahl) the streamwise-velocity non-uniformity accumulates indefinitely and grows linearly in time. Hultgren & Gustavsson (1981) observed that in the presence of viscosity an initially inviscid growth phase is followed by viscous decay.

In more recent years, while the linear lift-up phenomenon and its subsequent nonlinear evolution were being more and more clearly observed in numerical simulations (e.g. Henningson, Lundbladh & Johansson 1993, and papers referred to therein), the role of algebraic growth, including but not limited to the lift-up mechanism, was put into a new mathematical perspective by the method of pseudospectra (as described in the review by Trefethen *et al.* 1993, and references therein). In elementary calculus one is taught that the solution of a system of linear differential equations with constant coefficients is composed of a sum of exponentials, if the eigenvalues of the coefficient matrix are all distinct, or of exponentials multiplied by algebraic polynomials of time, if multiple eigenvalues occur. (A polynomial by itself can also occur, if the multiple eigenvalue happens to be zero.) One is also taught that this (sometimes called 'secular', a term borrowed from astronomy) polynomial behaviour is gradually approached as exponentials with similar exponents tend to each other. However, no secular behaviour takes place if the coefficient matrix is self-adjoint, and several matrices naturally occurring in physics are. Even though the general small-perturbation problem in fluid mechanics is well known not to be self-adjoint, for a long time it was more or less unconsciously assumed to behave almost as if it were. The non-coincidence of left and right eigenvectors was considered a mere technicality.

The method of pseudospectra, instead, provides a way to measure the departure from self-adjointness, and shows that when this departure is great, and in several fluid mechanics problems it is, the asymptotic exponential behaviour dictated by eigenvalue analysis may be overshadowed by a significant transient phase of algebraic growth, just as in the occurrence of lift-up, in some situations lasting long enough to make the exponential phase unobservable in practice. Numerical examples of this behaviour have been produced with reference to the time evolution of perturbations transiently imposed onto an otherwise steady base flow, both in actual fluid mechanics problems and in low-dimensional model systems (Bagget, Driscoll & Trefethen 1995). Mechanisms have been also proposed according to which the combination of an energetically small nonlinear mixing and energy-producing linear algebraic amplification might constitute the central feedback loop of turbulence generation (Boberg & Brosa 1988; Trefethen *et al.* 1993).

The three-dimensional boundary-layer instability analysed in the present paper adds two twists to the above framework. First, it is an instance of algebraic growth in space rather than in time. That is, it applies to the situation, more often encountered in experiments, in which a disturbance is applied at a specified upstream spatial location for all time rather than at a specified time throughout all of space. Secondly, it is an instance of algebraic growth (in particular, of the lift-up type) in a non-parallel base flow: a growing boundary layer. Non-parallelism has the peculiar consequence that, at least within the boundary-layer approximation (7), the algebraic growth goes on indefinitely even in the presence of viscosity, whereas in parallel flow the algebraic growth is always eventually followed by viscous decay (if the phenomenon remains linear up to that stage).

The final question is, of course: can the present theory describe bypass transition as seen in the experiments? At present we can only provide a tentative answer.

The most important parameter that the present theory should try to predict is the threshold where the u-component perturbation, extrapolated according to linear evolution, attains the same amplitude as the base flow, presumably a reasonable approximation of the threshold where transition is observed to occur in experiments. However, there are several reasons why this prediction can only be qualitative. One is, of course, that nonlinear phenomena take place once the amplitude of the perturbation becomes large. An additional difficulty is that, the driven δu perturbation being proportional to α, the maximum conversion of spanwise into longitudinal energy takes place near one end of the allowable α range, namely for $\alpha \approx 1/\delta$, where the approximation of replacing (8) by (7) becomes inaccurate. (Notice that this phenomenon is simultaneous with the onset of nonlinearities: just as at the beginning of the boundary layer the amplitude of the perturbation is small and later becomes comparable to that of the base flow, at the beginning of the boundary layer the local δ, which increases with x, will be small compared to $1/\alpha$, which is fixed by the oncoming disturbance, and only at the position where the instability becomes visible will $\alpha\delta$ be of order unity and (7) become inaccurate. But by this stage (8) themselves are insufficient and only the nonlinear equations (6) can be relied upon.) Therefore, all we can say is that maximum conversion takes place around a wavenumber which is of the order of $1/\delta$ in both the z- and y-directions, and that the corresponding amplification is proportional, through a presently unknown $O(1)$ constant, to αL, that is to L/δ, or to $R_\delta = (R_L)^{1/2}$ (in amplitude). In energy the amplification will be squared, that is to say, proportional to $(L/\delta)^2$ or R_δ^2 or R_L.

This means that, for instance, in order to provoke transition at a position corresponding to a Reynolds number $R_L = 5\times10^5$, as in the recent experiment on free-stream induced turbulence by Westin *et al.* (1994), one would have to inject a perturbation of energy $2\times10^{-6}U^2$ in the right wavenumber range of order $1/\delta$. Unfortunately, it is very difficult to say whether the turbulence-generating grid used by Westin *et al.* did just that, even though the authors took great care to create precisely characterized free-stream conditions. What seems apparent is that they did have sufficient energy, since the measured r.m.s. value of crossflow fluctuations upstream of the leading edge was $0.015U$, and therefore the total injected energy over all wavenumbers was $2\times10^{-4}U^2$ against an amplification of (the order of) 5×10^5; however, even though the paper gives detailed autocorrelation data, it is very difficult to say how much of this energy fell in 'the right wavenumber range' until we know more precisely where this range is. (It would make a lot of difference if it were, say, around $1/5\delta$ rather than around $1/\delta$).

8. Conclusions

A three-dimensional mode of instability of the Blasius boundary layer over a flat plate has been described, which results from the competition between inviscid algebraic growth and viscous dissipation. Whereas in a parallel flow the viscous dissipation eventually wins and converts the algebraic growth into exponential decay, in an expanding boundary layer the viscous dissipation only provides algebraic decay, and turns out to be insufficient to compensate the algebraic growth.

A distinguishing feature of this boundary-layer instability is its independence of the Reynolds number of the unpertubed flow. Such independence trivially follows from the fact that the phenomenon is governed by Prandtl's standard boundary-layer equations, where the Reynolds number has been scaled out. It follows that such an instability can be excited (by a large enough spanwise perturbation) however early in the development of the boundary layer, and therefore can provide a driving mechanism for those experimentally observed phenomena of early transition under moderately large disturbances that go under the name of bypass transition.

A qualitative comparison with experimentally observed orders of magnitude make the spatial boundary-layer instability described in the present paper a good candidate for the initial linear-amplification stage of bypass transition, opening the way to a better understanding of the origins of this phenomenon.

From a fundamental viewpoint, realizing that an instability can occur within the limits of the Prandtl approximation significantly affected the author's own insight about boundary-layer theory. In addition to allowing for the existence of a Reynolds-number-independent instability, the newfound capability of the boundary-layer equations to model at the least some instability, in a world where fluid-flow instabilities do happen, lends new confidence to their modelling power.

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Appendix A. The calculation of Libby–Fox mode amplitudes from incident free-stream perturbations

The procedure proposed by Libby & Fox (1964) for the calculation of initial mode amplitudes begins with the sturm–Liouville equation

$$[(F_0'^3/F_0'')H']' + (rF_0'^4/F_0'' - F_0F_0'^2)H = 0 \tag{A 1}$$

for the auxiliary function $H = (N/F_0')'$ and consists of applying the orthogonality relation appropriate to this equation, i.e.

$$\int_0^\infty H_k H_l (F_0'^4/F_0'')\,\mathrm{d}\eta = C_k\,\delta_{kl}, \tag{A 2}$$

where

$$C_k = \int_0^\infty (H_k)^2 (F_0'^4/F_0'')\,\mathrm{d}\eta. \tag{A 3}$$

Therefore, with an initial condition $\delta F(x_i, \eta)$ given at a certain non-zero abscissa x_i, the perturbation solution was written by Libby & Fox as a mode expansion of the form

$$\delta F = \sum_{k=1}^{\infty} A_k \left(\frac{x_i}{x}\right)^{r_k} N_k(\eta), \tag{A 4}$$

where N_k denotes the kth eigensolution of (4), normalized with the condition $N_{\eta\eta}(0) = 1$, and r_k is the corresponding eigenvalue. The mutual orthogonality of the functions $H_k = (N_k/F_0')'$ allows the coefficients A_k to be determined as

$$A_k = C_k^{-1} \int_0^{\infty} [\delta F(x_i, \eta)/F_0']' \, [N_k/F_0']' \, (F_0'^4/F_0'') \, \mathrm{d}\eta. \tag{A 5}$$

Libby & Fox's analysis of the modal expansion stops here (they then proceed to apply their technique to the solution of some specific incompressible and compressible boundary-layer problems). It may thus appear that, owing to the singularity of the similarity variables at $x = 0$ (A 4)–(A 5) only apply when the initial condition is given at a non-zero x_i. However, if (A 4) is rewritten in terms of the coefficients $a_k = A_k x_i^{r_k}$, as

$$\delta F = \sum_{k=1}^{\infty} a_k x^{-r_k} N_k(\eta) \tag{A 6}$$

it becomes evident that the coefficients a_k must be independent of the abscissa x_i that is chosen for the purpose of calculating them. In other words, the expressions

$$a_k = x_i^{r_k} C_k^{-1} \int_0^{\infty} [\delta F(x_i, \eta)/F_0']' \, [N_k/F_0']' \, (F_0'^4/F_0'') \, \mathrm{d}\eta, \tag{A 7}$$

when considered as functions of x_i, are constants of the motion. Since the coefficients a_k are constant, their limits for $x_i \to 0$ trivially exist. Therefore, the expansion coefficients may as well be calculated from the initial profile of the free-stream perturbation given at $x_i = 0$.

In order to cast (A 7) into an explicit limiting form, let us first integrate by parts to obtain

$$a_k = x_i^{r_k} C_k^{-1} \int_0^{\infty} \delta F(x_i, \eta) \, L_k(\eta) \, \mathrm{d}\eta, \tag{A 8}$$

where

$$L_k = -\frac{1}{F_0'} \left[\frac{(N_k/F_0')' \, F_0'^4}{F_0''}\right]'. \tag{A 9}$$

Now, on restoring the original variables x, y and $\delta\psi(x, y) = x^{1/2} \delta F(x, \eta)$, (A 7) can also be written as

$$a_k = C_k^{-1} \int_0^{\infty} \delta\psi(x_i, y) \, L_k(y/x_i^{1/2}) \, x_i^{(r_k - 1)} \, \mathrm{d}y. \tag{A 10}$$

The limit for $x_i \to 0$ of $L_k(y/x_i^{1/2}) \, x_i^{(r_k-1)}$ is non-zero and finite (as it must be, since a_k is independent of x_i); for, we can in this limit replace $L_k(\eta)$ by its behaviour for large η, which is $L_k \sim \eta^{2(r_k-1)}$ as can be obtained either directly from a WKB analysis of (4) or from the method of Appendix B, and therefore (A 10) becomes

$$a_k = \frac{c_k}{C_k} \int_0^{\infty} \delta\psi(0, y) \, y^{(2r_k - 2)} \, \mathrm{d}y \tag{A 11}$$

or equivalently

$$a_k = -\frac{c_k}{(2r_k-1)\,C_k}\int_0^{\infty} \delta u(0,y)\,y^{(2r_k-1)}\,\mathrm{d}y \tag{A 12}$$

with c_k denoting a new constant.

This is the required expression for the expansion coefficients in terms of the initial perturbation profile given at $x_i = 0$. The first ten values of the coefficients c_k and C_k are given in table 1. The procedure used to estimate the asymptotic coefficients c_k is detailed in Appendix B.

Appendix B. Limiting behaviour of the eigenfunctions for large η

In order to derive an asymptotic approximate solution of (4), we can replace F_0 by its large-η behaviour, namely

$$F_0 = \eta - a + \text{exponentially small terms}, \tag{B 1}$$

where, from the numerical solution of Blasius' equation (2), $a = 1.720787657$. Equation (4) thus reduces to

$$N_{\eta\eta\eta} + \tfrac{1}{2}(\eta - a)\,N_{\eta\eta} + rN_{\eta} = 0, \tag{B 2}$$

which can be recognized as a Hermite equation in the unknown N_η. A solution of (B 2) can be expressed, to within an arbitrary multiplicative constant, through the Laplace-transform integral

$$N_\eta = \int_{-i\infty}^{i\infty} p^{2r-1}\,\mathrm{e}^{p^2+(\eta-a)p}\,\mathrm{d}p. \tag{B 3}$$

The solution branch that is finite at positive infinity is selected when the complex path of integration runs on the left side of the branch point that for general values of r appears at the origin of the p-plane. (For semi-integral values of r the Hermite function reduces to a Hermite polynomial times a Gaussian exponential, but this is not the present case.) If, in particular, we move the integral to the steepest-descent path through the saddle point $p = -(\eta - a)/2$, by writing $p = -(\eta-a)/2 + \mathrm{i}p'$, (B 3) may be recast as

$$N_\eta \sim \left(\frac{\eta - a}{2}\right)^{2r-1} \mathrm{e}^{-(\eta-a)^2/4}\, C(\eta - a), \tag{B 4}$$

where

$$C(\eta - a) = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\left[1 - \frac{2\mathrm{i}p'}{\eta - a}\right]^{2r-1} \mathrm{e}^{-p'^2}\,\mathrm{d}p', \tag{B 5}$$

and $C(\eta - a) \to 1$ for $\eta \to \infty$. Therefore (B 4) represents the asymptotic behaviour of $N_\eta(\eta)$ for $\eta \to \infty$. According to (B 1), this may also be equivalently written as

$$N_\eta \sim (\tfrac{1}{2}F_0)^{2r-1}\,\mathrm{e}^{-F_0^2/4}\,[C(F_0) + \text{exponentially small terms}]. \tag{B 6}$$

In Appendix A we have used the asymptotic large-η behaviour of the quantity L_k defined by (A 9). On using (B 1) again, we can recast (A 9) as

$$L_k = -(N_k'/F_0'')' + \text{exponentially small terms}, \tag{B 7}$$

i.e. from (B 4) with $C(\eta - a)$ replaced by 1 and from $F_0'' \sim \mathrm{e}^{-(\eta-a)^2/4}$,

$$L_k \approx c_k\,\eta^{2r_k-2}, \tag{B 8}$$

which is the result we needed in the derivation of (A 11).

The coefficient c_k has to be computed from the numerical solution of the full equation (4), since it depends critically on the boundary conditions given at zero and not only on the behaviour at infinity.

A straightforward procedure would be to calculate the ratio L_k/η^{2r_k-2} for increasing values of η until this ratio approaches a recognizable limit. However, this naive method is bound to give very poor results. This is because the relative error within which the asymptotic behaviour (B 8) is approached is $O(1/\eta)$, and continuing the numerical solution to very large values of η, in addition to being expensive, is also inaccurate because of the superposition of truncation and roundoff errors over the rapidly decreasing $O[e^{-(\eta-a)^2/4}]$ solution.

A much better estimate of c_k can be obtained from the ratio between a numerical solution of (4) and an exponentially accurate asymptotic approximation of the same solution. If in equation (B 6) the function $C(F_0)$ is retained as a corrective coefficient rather than replaced with its limit, we can write

$$\frac{N'_k}{F''_0} = -\frac{c_k}{2r_k-1} F_0^{2r_k-1}\, C(F_0) + \text{exponentially small terms}, \tag{B 9}$$

so that c_k can now be obtained with much greater precision if $C(F_0)$ is known. (In other words, rather than estimating c_k from the ratio between the numerical solution and its leading asymptotic behaviour, we are now estimating it from the ratio between the numerical solution of the complete equation and the solution of its asymptotic form (B 2).) The only remaining task is to calculate the solution of the Hermite equation (B 2), or equivalently the corrective coefficient $C(F_0)$. Whereas asymptotic power expansions of Hermite functions are available in analytical form in textbooks, the precision available with those expansions is still limited by their very asymptotic character. On the other hand an efficient method, which can be used to evaluate with arbitrary precision any special function for which an integral representation is known, was introduced by Luchini & Bassano (1991). It consists quite simply of calculating the steepest-descent complex integral (in the present case, (B 5)) by numerical quadrature. In fact, for the unbounded integral of an analytic function that decays exponentially at infinity almost any discretization method yields exponential accuracy, even the lowly constant-step, constant-weight summation. The choice of the steepest-descent path, or of one close to it, ensures a smooth non-oscillatory behaviour of the integrand so that even a moderate number of sampling points is sufficient to obtain a high accuracy.

The values reported in table 1 were obtained by calculating N_k and F_0, simultaneously with their derivatives, through a fourth-order Runge–Kutta shooting method and $C(F_0)$ by midpoint constant-step, constant-weight quadrature of (B 5). Using 30 sampling points in the numerical quadrature (from $p' = 0$ to $p' = 6$ with step 0.2) yielded the accuracy shown. A similar procedure was used for table 2.

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